

Inertial modes of slowly rotating isentropic stars

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ABSTRACT

We investigate inertial mode oscillations of slowly and uniformly rotating, isentropic, Newtonian stars. Inertial mode oscillations are induced by the Coriolis force due to the star's rotation, and their characteristic frequencies are comparable with the rotation frequency Ω of the star. So called r -mode oscillations form a sub-class of the inertial modes. In this paper, we use the term “ r -modes” to denote the inertial modes for which the toroidal motion dominates the spheroidal motion, and the term “inertial modes” to denote the inertial modes for which the toroidal and spheroidal motions have comparable amplitude to each other. Using the slow rotation approximation consistent up to the order of Ω^3 , we study the properties of the inertial modes and r -modes, by taking account of the effect of the rotational deformation of the equilibrium on the eigenfrequencies and eigenfunctions. The eigenfrequencies of the r -modes and inertial modes calculated in this paper are in excellent agreement with those obtained by Lindblom et al (1999) and Lockitch & Friedman (1998). We also estimate the dissipation timescales due to the gravitational radiation and several viscous processes for polytropic neutron star models. We find that for the inertial modes, the mass multipole gravitational radiation dominates the current multipole radiation, which is dominating in the case of the r -modes. It is also found that the inertial mode instability is more unstable than previously reported by Lockitch & Friedman (1998), and survives the viscous damping processes relevant in neutron stars.

Subject headings: instabilities — stars: neutron — stars: oscillations — stars: rotation

1. Introduction

Since Andersson (1998) suggested and Friedman & Morsink (1998) analytically verified that the r -modes of rotating stars are unstable due to the gravitational radiation reaction, much attention has been paid to oscillation modes of rotating neutron stars because of their possible importance in a field of astrophysics. Lindblom et al. (1998) argued that due to this instability, the maximum angular rotation velocity of hot young neutron stars is strongly restricted. Owen et al. (1998) also suggested that the gravitational radiation emitted from hot young neutron stars due to the r -mode instability is expected to be one of the potential sources for the gravitational

wave detectors, e.g. LIGO. Following these earlier investigations on the r -mode instability, there have appeared a number of papers which study the properties of the r -modes of rotating stars (e.g., Kojima 1998, Anderson et al 1998, Kokkotas & Stergioulas 1998, Lindblom & Iser 1998, Beyer & Kokkotas 1999, Lindblom et al 1999, Kojima & Hosonuma 1999).

It is well known that the r -modes form a sub-class of the modes called “inertial modes”, for which the restoring force is the Coriolis force and the characteristic frequency is comparable to the angular rotation frequency Ω of the star (e.g., Greenspan 1964). As the prominent characteristics of the r -modes, we know that the toroidal motion dominates the spheroidal motion in the velocity field, and that the frequency ω observed in the corotating frame of the star is given by $\omega = 2m\Omega/(l(l+1))$ in the limit of $\Omega \rightarrow 0$, where m and l are the indices of a spherical harmonic function Y_l^m representing the toroidal component of the velocity field (e.g., Papaloizou & Pringle 1978, Provost et al 1981, Saio 1982). As shown by Lee et al (1992), however, the inertial modes in general have the spheroidal component of the velocity field comparable with the toroidal component and do not have any analytical formula to give ω/Ω in the limit of $\Omega \rightarrow 0$, except for the case of the Maclaurin spheroids (Lindblom & Iser 1998; see also Bryan 1889). In this paper, to distinguish these two classes of inertial modes, we use the term of “inertial modes” to refer to the inertial modes with the comparable spheroidal and toroidal components of the velocity field, and the term of “ r -modes” to refer to the inertial modes whose toroidal component of the velocity field is dominating the spheroidal component.

Very recently, Lockitch & Friedman (1998) calculated the inertial modes, assuming an ordering law given by

$$\begin{aligned} S_l &\sim O(1), & H_l &\sim O(1), & iT_{l'} &\sim O(1), \\ \delta\rho &\sim O(\Omega^2), & \delta p &\sim O(\Omega^2), & \delta\Phi &\sim O(\Omega^2), & \sigma &\sim O(\Omega), \end{aligned} \tag{1}$$

where S_l and H_l are the spheroidal components and $iT_{l'}$ is the toroidal component of the displacement vector, and $\delta\rho$, δp , and $\delta\Phi$ are the Eulerian perturbations of the density, pressure, and gravitational potential, and σ is the oscillation frequency observed in an inertial frame. For r -modes, we may assume an ordering law given as $\sigma \sim O(\Omega)$, $iT_{l'} \sim O(1)$, and $S_l \sim H_l \sim \delta\rho \sim \delta p \sim \delta\Phi \sim O(\Omega^2)$. Lockitch & Friedman (1998) showed that a number of inertial modes are unstable because of the gravitational radiation reaction, just as r -modes are, and estimated the dissipation timescales due to the gravitational radiation and some viscous processes for slowly rotating polytropic neutron star models, where they made use of eigenfrequencies and eigenfunctions for slowly rotating Maclaurin spheroids to classify the inertial modes. Notice that Lindblom & Iser (1998) obtained analytically eigenfrequencies and eigenfunctions of the inertial mode for the Maclaurin spheroids (see also Bryan 1889). Lockitch & Friedman (1998) and Lindblom & Iser (1998) called the inertial modes “rotation modes” or “generalized r -modes”, which are exactly the same as the inertial modes that will be discussed in this paper.

In this paper, we study the inertial modes and r -modes of slowly rotating stars, by employing a different numerical approach to the problem (Lee & Saio 1986, Lee 1993) from the calculations

by, e.g., Lindblom et al. (1999), and Lockitch & Friedman (1998). Newtonian gravity, uniform rotation, and isentropic structure of the equilibrium are assumed for adiabatic perturbations in this paper. The effects of the rotational deformation of the equilibrium structure on the eigenfunctions and eigenfrequencies are included in the analysis, where the formulation by Lee (1993) (see also Lee & Saio 1986) has been improved so that the eigenmodes are consistent up to the order of Ω^3 . The plan of this paper is as follows. In §2, we describe briefly the basic equations for the equilibrium configurations of slowly rotating polytropes. In §3, we present the formalism to calculate normal modes of slowly rotating stars. In §4, the properties of the eigenfrequencies and eigenfunctions of the inertial modes are discussed. In §5, we examine the stability of simple neutron star models against the inertial modes and the r -modes, computing their growth or damping timescales due to the gravitational radiation reaction and some viscous damping processes. §6 is for discussions and conclusions.

2. Equilibrium State

We consider oscillations of an uniformly rotating, isentropic, Newtonian star. The structure of an equilibrium state is determined by the hydrostatic equation, the Poisson equation, and the equation of state:

$$\nabla_i p = -\rho \nabla_i \Psi, \quad (2)$$

$$\nabla_i \nabla^i \Phi = 4\pi G \rho, \quad (3)$$

$$p = p(\rho), \quad (4)$$

where Ψ is the effective potential defined by

$$\Psi = \Phi - \frac{1}{3} \Omega^2 r^2 \{1 - P_2(\cos \theta)\}, \quad (5)$$

and Ω is the angular rotation frequency of the star, constant for uniform rotation. Here we use spherical polar coordinates (r, θ, ϕ) , and $P_l(\cos \theta)$ denotes the Legendre polynomial, and ∇_i is the covariant derivative with respect to the quantity followed.

In this investigation, assuming slow rotation, we apply the Chandrasekhar-Milne expansion (see, e.g, Tassoul 1978) to equations (2)–(5). In this technique the effects of the centrifugal force and the equilibrium deformation are treated as small perturbations to a non-rotating spherically symmetric star. The small expansion parameter due to rotation is chosen as $\bar{\Omega} = \Omega(R^3/GM)^{1/2}$, where R and M are the radius and the mass of the non-rotating star, respectively. To the lowest order effects of the centrifugal force, the effective potential Ψ can be expanded as follows:

$$\Psi(r, \theta) = \Psi_0(r) - 2R^2\Omega^2 [\psi_0(r/R) + A_2\psi_2(r/R)P_2(\cos \theta) + O(\bar{\Omega}^2)], \quad (6)$$

where $\Psi_0(r)$ is the potential of the non-rotating star. The functions $\psi_0(x)$ and $\psi_2(x)$ are determined by solving the following ordinary differential equations:

$$\frac{1}{x^2} \frac{d}{dx} \left(x^2 \frac{d\psi_0}{dx} \right) = k(x)\psi_0(x) + 1, \quad (7)$$

$$\frac{1}{x^2} \frac{d}{dx} \left(x^2 \frac{d\psi_2}{dx} \right) = \left(k(x) + \frac{6}{x^2} \right) \psi_2(x), \quad (8)$$

$$k(x) = 4\pi G R^2 \frac{d\rho_0}{d\Psi_0}, \quad (9)$$

where $x = r/R$ and ρ_0 is the density of the non-rotating star. For simplicity, we consider a sequence of slowly rotating stars whose central density is the same as that of the non-rotating star. From this condition, together with the regularity condition at the center of the star, the boundary conditions of equations (7) and (8) at the origin are given as

$$\psi_i(0) = 0, \quad \frac{d\psi_i(0)}{dx} = 0, \quad (10)$$

where $i = 0, 2$. From the boundary conditions at the stellar surface, the constant A_2 is determined as

$$A_2 = -\frac{5}{6 (3\psi_2(1) + d\psi_2(1)/d \ln x)}. \quad (11)$$

3. Perturbation Equations

In the perturbation analysis, we introduce a parameter a that is constant on a distorted effective potential surface. In practice, the parameter a is defined such that

$$\Psi(r, \theta) = \Psi_0(a). \quad (12)$$

With this parameter a , the distorted equi-potential surface may be given by

$$r = a\{1 + \epsilon(a, \theta, \phi)\}. \quad (13)$$

By using equations (6), (12) and (13), we obtain the explicit expression for the function $\epsilon(a, \theta)$ up to the order of $\bar{\Omega}^2$:

$$\epsilon(a, \theta) = \alpha(a) + \beta(a)P_2(\cos \theta), \quad (14)$$

where

$$\alpha(a) = \frac{2c_1 \bar{\Omega}^2 \psi_0(x)}{x^2}, \quad (15)$$

$$\beta(a) = \frac{2c_1 \bar{\Omega}^2 A_2 \psi_2(x)}{x^2}. \quad (16)$$

Here, $c_1 = (a/R)^3/(M(a)/M)$, and $M(a)$ denotes the mass inside the a -constant surface.

Hereafter, we employ the parameter a as the radial coordinate, instead of the polar radius coordinate r . In this coordinate system (a, θ, φ) , the metric tensor is written by

$$ds^2 = (1 + 2\epsilon)(da^2 + a^2 d\theta^2 + a^2 \sin^2 \theta d\varphi^2) + 2a\epsilon_{,a} da^2 + 2a\epsilon_{,\theta} da d\theta + O(\bar{\Omega}^4), \quad (17)$$

where a comma denotes a partial derivative with respect to the variables followed. Note that in this frame the pressure, density and effective potential of a rotating star depend only on the one variable a , although the orthogonality of the basis vectors is lost.

The governing equations of nonradial oscillations of a rotating star are obtained by linearizing the basic equations used in fluid mechanics. Since the equilibrium state is assumed to be stationary and axisymmetric, the perturbations may be represented by a Fourier component proportional to $e^{i(\sigma t + m\varphi)}$, where σ is the frequency observed in an inertial frame and m is the azimuthal quantum number. The continuity equation may be linearized to be

$$\delta\rho = -\nabla_i(\rho\xi^i), \quad (18)$$

where ξ^i is the Lagrangian displacement vector, and we have made use of $\delta v^i = i(\sigma + m\Omega)\xi^i \equiv i\omega\xi^i$ with ω being the oscillation frequency observed in the corotating frame of the star. The linearized Euler's equation is

$$-\omega^2\xi_i + \nabla_i\left(\frac{\delta p}{\rho} + \delta\Phi\right) + A_i\frac{\delta p}{\rho} + \xi^j A_j \frac{1}{\rho}\nabla_i p + 2i\omega\Omega\xi^j\nabla_j\varphi_i = 0, \quad (19)$$

where φ^i is the rotational Killing vector, with which the 3-velocity of the equilibrium fluid of a rotating star is given as $v^i = \Omega\varphi^i$. The last term on the left-hand-side of equation (19) comes from the Coriolis force, and its explicit components in the metric (17) are given by

$$\xi^j\nabla_j\varphi_a = -(1 + 2\epsilon + a\epsilon_{,a})a\sin^2\theta\xi^\varphi + O(\bar{\Omega}^4), \quad (20)$$

$$\xi^j\nabla_j\varphi_\theta = -\{(1 + 2\epsilon)\cos\theta + \epsilon_{,\theta}\sin\theta\}a^2\sin\theta\xi^\varphi + O(\bar{\Omega}^4), \quad (21)$$

$$\xi^j\nabla_j\varphi_\varphi = -(1 + 2\epsilon + a\epsilon_{,a})a\sin^2\theta\xi^a - \{(1 + 2\epsilon)\cos\theta + \epsilon_{,\theta}\sin\theta\}a^2\sin\theta\xi^\theta + O(\bar{\Omega}^4). \quad (22)$$

(Note that Lee (1993) ignored the terms of ϵ , $\epsilon_{,a}$, and $\epsilon_{,\theta}$ in $\xi^j\nabla_j\varphi_i$, as a result of which the eigenmodes obtained are consistent only up to the order of $\bar{\Omega}^2$. By retaining these terms in the analysis, the eigenmodes calculated in this paper are consistent up to the order of $\bar{\Omega}^3$ for slow rotation.) The perturbed Poisson equation is given by

$$\nabla_i\nabla^i\delta\Phi = 4\pi G\delta\rho. \quad (23)$$

For adiabatic oscillations, we have

$$\delta p = \Gamma p \left(\frac{\delta\rho}{\rho} + \xi^i A_i \right) \quad (24)$$

where Γ is the adiabatic index given by

$$\Gamma = \left(\frac{\partial \ln p}{\partial \ln \rho} \right)_{ad}, \quad (25)$$

and A_i is the generalized Schwarzschild discriminant defined by

$$A_i = \frac{1}{\rho}\nabla_i\rho - \frac{1}{\Gamma p}\nabla_i p. \quad (26)$$

In the case of isentropic stars, the 1-form A_i vanishes exactly.

Physically acceptable solutions of the linear differential equations are obtained by imposing boundary conditions at the inner and outer boundaries of the star. The inner boundary conditions are the regularity condition of the perturbed quantities at the stellar center. The outer boundary conditions at the surface of the star are $\Delta p/\rho = 0$ and the continuity of the perturbed gravitational potential at the surface to the solution of $\nabla_i \nabla^i \delta\Phi = 0$ which vanishes at infinity.

In order to solve the system of partial differential equations given above, we employ a series expansion in terms of spherical harmonics to represent the angular dependence of the perturbed quantities. The Lagrangian displacement vector, ξ^i , is expanded in terms of the vector spherical harmonics as

$$\xi^a = \sum_{l \geq |m|}^{\infty} a S_l(a) Y_l^m(\theta, \varphi) e^{i\sigma t}, \quad (27)$$

$$\xi^\theta = \sum_{l, l' \geq |m|}^{\infty} \left\{ H_l(a) Y_l^m{}_{,\theta} + T_{l'}(a) \frac{1}{\sin \theta} Y_{l'}^m{}_{,\varphi} \right\} e^{i\sigma t}, \quad (28)$$

$$\xi^\varphi = \frac{1}{\sin^2 \theta} \sum_{l, l' \geq |m|}^{\infty} \left\{ H_l(a) Y_l^m{}_{,\varphi} - T_{l'}(a) \sin \theta Y_{l'}^m{}_{,\theta} \right\} e^{i\sigma t} \quad (29)$$

(Regge & Wheeler 1957, see also Thorne 1980). The perturbed scalar quantity such as $\delta\Phi$ is expressed as

$$\delta\Phi = \sum_{l \geq |m|}^{\infty} \delta\Phi_l(a) Y_l^m(\theta, \varphi) e^{i\sigma t}. \quad (30)$$

Substituting the perturbations into the linearized basic equations (18), (19), (23) and (24), we obtain an infinite system of coupled linear ordinary differential equations for the expansion coefficients. The details are given in the Appendix.

For numerical calculations, we truncate the infinite set of linear ordinary differential equations to obtain a finite set by discarding all the expansion coefficients associated with l higher than l_{max} . This truncation determines the number of the expansion coefficients kept in the spherical harmonic expansion of each perturbed quantity. We denote this number as k_{max} . Note that the number k_{max} is equivalent to the dimension of the column vectors \mathbf{y}_i , introduced in the Appendix, for $i = 1-4$. Our basic equation therefore becomes a system of $4 \times k_{max}$ -th order ordinary differential equations, which, together with the boundary conditions, are numerically solved as an eigenvalue problem of $\omega \equiv \sigma + m\Omega$ by using a Henyey type relaxation method (e.g., Unno et al . 1989, see also Press et al. 1992).

For the number k_{max} of the expansion coefficients kept in the eigenfunctions, we employ $k_{max}=5$ or 6, values that are found to be large enough for the inertial modes calculated in this paper. Note that if we consider higher radial-order modes, with many radial nodes of the eigenfunctions, than the modes obtained in this study, the effects of truncation can be more serious, as shown by Lee et al. (1992).

When $\Omega = 0$, nonradial oscillation modes are classified into two decoupled sets, called the “polar” parity mode (or the “spheroidal” mode) and the “axial” parity mode (or the “toroidal” mode). Although these modes have been traditionally called “even” and “odd” parity modes, respectively, we will use these terms for different meanings in this paper (see below). When $\Omega \neq 0$, both parity modes are mixed, and an oscillation mode contains contributions from the polar and axial parity modes.

Since the equilibrium state of a rotating star is invariant under the parity transformation defined by $\theta \rightarrow \pi - \theta$, and is symmetric with respect to the equator, the linear perturbations have definite parity for that transformation. In this paper, a set of modes whose scalar perturbations such as $\delta\Phi$ are symmetric with respect to the equator is called “even” modes, while a set of modes whose scalar perturbations are antisymmetric with respect to the equator is called “odd” modes (see, e.g., Berthomieu et al. 1978). For positive integers $k = 1, 2, 3, \dots$, we have $l = |m| + 2k - 2$ and $l' = l + 1$ for even modes, and $l = |m| + 2k - 1$ and $l' = l - 1$ for odd modes, where the symbols l and l' have been used to denote the spheroidal components and toroidal components of the displacement vector ξ^i , respectively (see equations (27)–(30)). Notice that in Lockitch & Friedman (1998), the terms “even” and “odd” modes are used to denote the “polar-led hybrids” and “axial-led hybrids” modes, respectively.

4. The Eigenvalues and Eigenfunctions of Inertial Modes

We compute the eigenvalues and eigenfunctions of inertial modes and r -modes of rotating polytropic stars with accuracy up to the order of $\bar{\Omega}^3$. In this study, we concentrate our attention to the case of isentropic stars, for which the generalized Schwarzschild discriminant A_i vanishes exactly and the equilibrium state is marginally stable against convection. In this case, the adiabatic index is given by using the polytropic index n as

$$\Gamma = \frac{d \ln p}{d \ln \rho} = \frac{n+1}{n}. \quad (31)$$

Since the rotational deformation is of the order of $\bar{\Omega}^2$, the effects on the frequencies ($\propto \bar{\Omega}$) of the inertial modes and r -modes appear as terms of the order of $\bar{\Omega}^3$. For sufficiently small values of $\bar{\Omega}$, therefore, we may expand the eigenfrequency ω of the inertial modes and r -modes of a rotating star in powers of $\bar{\Omega}$ as

$$\frac{\omega}{\Omega} = \kappa_0 + \kappa_2 \bar{\Omega}^2 + O(\bar{\Omega}^4). \quad (32)$$

In this study, the expansion coefficients κ_0 and κ_2 are obtained by fitting the eigenfrequencies computed for a mode at several small values of $\bar{\Omega}$.

For nonradial p -, f -, and g -modes of a non-rotating or sufficiently slowly rotating star, we may specify the values of l and m without any ambiguity and, by counting the number of radial nodes of the eigenfunctions, we may order the modes without any confusion (e.g., Unno et al. 1989). But, to our knowledge, no well-established classification scheme for inertial modes of a rotating star

exists, except for the case of the Maclaurin spheroids, for which exact eigensolutions are known (Bryan 1889; Lindblom & Ipser 1998). As shown by Bryan (1889), eigenmodes of the Maclaurin spheroid can be characterized by the eigenvalue ω/Ω and the angular quantum numbers l_0 and m of the associated Legendre functions of the spheroidal coordinate defined on a surface form of the spheroid. The functional forms of the eigenfunctions in the spheroidal coordinate, which depends on the eigenvalue ω/Ω , are specified by l_0 and m . Since the the spheroidal coordinate employed is dependent on ω/Ω , the functional forms in the actual spatial coordinate are different for different eigenvalues ω/Ω for given l_0 and m . In fact, Lindblom & Ipser (1998) have shown that, for given l_0 and m , the eigenvalue κ_0 for inertial modes of the Maclaurin spheroid can be determined, to lowest order of $\bar{\Omega}$, by solving the algebraic equation given by

$$m \frac{d^m}{dx^m} P_{l_0}(\kappa_0/2) + \left(\frac{\kappa_0}{2} - 1 \right) \frac{d^{m+1}}{dx^{m+1}} P_{l_0}(\kappa_0/2) = 0, \quad (33)$$

where $m \geq 0$ is assumed. They also have shown that

$$-2 < \kappa_0 < 2, \quad (34)$$

and that the number of the roots of equation (33) is equal to $l_0 - m$.

In this paper, as a classification scheme for inertial modes of uniformly rotating polytropic stars with $n \neq 0$, we follow the scheme used for the inertial modes of the Maclaurin spheroid, and we pick up the inertial modes that are similar in the mode character to those of the Maclaurin spheroid, labeling them with the quantum numbers l_0 and m . In practice, the value of l_0 is determined by examining the properties of the eigenfunctions, such as the number of radial nodes of the expansion coefficients, and the number of the dominant expansion coefficients of the eigenfunctions. A similar procedure was employed by Lockitch & Friedman (1998). Notice that our definition of l_0 is not the same as that of Lockitch & Friedman (1998), but the same as that of Lindblom & Ipser (1998).

In Table 1 we tabulate the eigenvalues (κ_0, κ_2) for inertial modes and r -modes of a polytropic model with $n = 1$. As expected from the results for the Maclaurin spheroid, the value of $l_0 - |m|$ is odd for odd modes and even for even modes. The modes which satisfy the condition $\sigma(\sigma + m\Omega) < 0$ are unstable to the gravitational radiation reaction (see the next section), and are marked with an asterisk *. We find that a number of inertial modes are unstable to the gravitational radiation reaction, as first shown by Lockitch & Friedman (1998). The modes with $l_0 - |m| = 1$ are r -modes and the frequency to lowest order in Ω is given by $\kappa_0 = 2/(|m| + 1)$, which is independent of the structure of the star. Only the r -modes with $l_0 - 1 = |m|$, for which the toroidal component $iT_{l'}$ associated with $Y_{l'=|m|}^m$ is dominating, are found in isentropic stars, as suggested by Saio (1982). Table 1 also shows that, although r -modes are all retrograde, there are both prograde and retrograde inertial modes.

In Table 2, the eigenfrequencies (κ_0, κ_2) of inertial modes and r -modes with $m = 2$ are tabulated for several values of the polytropic index n , to see the dependence of the eigenfrequencies

on the equation of state, where the eigenfrequencies for the case of $n = 0$ are obtained by using the results by Lindblom & Ipser (1998). As shown by Table 2, the eigenvalues (κ_0, κ_2) of the inertial modes depend on the polytropic index n , while only the eigenvalue κ_2 of the r -modes is dependent on the index n .

Let us compare our calculations to those by Lindblom et al (1999) and by Lockitch & Friedman (1998). We find that the r -mode eigenfrequencies (κ_0, κ_2) given in Table 1 for $m = 2$ and 3 are in good agreement with those given in table 1 of Lindblom et al. (1999). Note that the normalization applied to κ_2 employed by Lindblom et al. (1999) is different, by a factor $4/3$, from that employed in this paper. We also find that the inertial mode frequencies κ_0 given in Tables 1 and 2 are in good agreement with those given in the table 6 of Lockitch & Friedman (1998), who do not give any number for κ_2 because of their neglect of the effects of the rotational deformation.

In Figures 1 to 3, we show the several expansion coefficients $S_l(a/R)$, $H_l(a/R)$ and $iT_{l'}(a/R)$ for four different $m = 2$ inertial modes of a polytropic model with $n = 1$, where the amplitude normalization is given at the surface of the model by $iT_{|m|+1} = 1$ for even modes and $iT_{|m|} = 1$ for odd modes. In Figure 1, the first two expansion coefficients are plotted, versus the fractional radial coordinate defined as a/R , for the even parity inertial modes with $m = 2$ and $l_0 - |m| = 2$. The solid curves give the expansion coefficients corresponding to the mode with $\kappa_0 = 1.100$, while the dotted curves correspond to the mode having $\kappa_0 = -0.557$. This figure shows that only the first expansion coefficients are dominating for both axial and polar components of the displacement vector. In Figures 2 and 3, the first three expansion coefficients of the spheroidal and toroidal components of the displacement vector are plotted against the fractional radial coordinate a/R for the unstable even parity inertial modes with $m = 2$ and $l_0 - |m| = 4$. Figures 2 and 3 correspond to the mode with $\kappa_0 = 1.520$ and with $\kappa_0 = 0.863$, respectively. The figures show that only the first two expansion coefficients of the eigenfunctions have dominant amplitude. Figures 1 to 3 may confirm that the series expansion of the eigenfunctions in terms of vector spherical harmonics converges quickly for the inertial modes calculated here. We note that the behavior of the eigenfunctions is very similar to that of the corresponding modes of the Maclaurin spheroid (see also Lockitch & Friedman 1998). If we consider the modes associated with given l_0 and m , the number of nodes of the dominant expansion coefficients in the radial direction is the same for the modes with different eigenvalues κ_0 . This means that the number of radial nodes of the eigenfunctions is not necessarily a good quantum number to label the modes with different eigenvalues κ_0 but with the same l_0 and m .

Figures 1 to 3 also show that for inertial modes of a slowly rotating isentropic star, the toroidal expansion coefficients $iT_{l'}$ has comparable amplitude to the spheroidal expansion coefficients S_l and H_l in the interior of the star. It is also shown that since the eigenfunctions S_l have vanishing amplitude at the surface, the fluid motion is almost tangential to the stellar surface.

In Table 3, we tabulate the number of radial nodes of the dominant expansion coefficients of ξ^i of the r -modes and inertial modes for several values of $l_0 - |m|$. Note that the content of

Table 3 is the same for $m = 1, 2, 3$. The number of the dominant expansion coefficients of the eigenfunctions $iT_{l'}$ may be given as the maximum positive integer k that satisfies $2k - 1 \leq l_0 - |m|$. Also the number of the dominant expansion coefficients of the spheroidal components, S_l and H_l , may be given as k for the even parity modes and as $k - 1$ for the odd parity modes. Although we cannot give any proof in general, Table 3 may show that the number of radial nodes of the dominant expansion coefficients $iT_{l'}$ with the highest harmonic index l' is zero, and that the number of radial nodes of the dominant expansion coefficients $iT_{l'}$ with the lowest harmonic index l' is given by the maximum positive integer k which satisfies $2k + 1 \leq l_0 - |m|$.

5. Dissipation Timescales

As mentioned in the previous section, some of the inertial modes calculated in this paper are unstable to the gravitational radiation reaction in the sense that the frequency σ satisfies the condition $\sigma(\sigma + m\Omega) < 0$ (see, equations (43) and (44)). However, since in neutron stars there are some possible viscous and dissipative processes which tend to suppress the instability, to decide whether the rotating neutron star is really unstable due to the gravitational radiation reaction, we need to know that the instability is strong enough to survive the dissipative processes.

The effects of the gravitational radiation reaction and viscous processes on the r -modes have already been studied by a number of authors (Lindblom et al 1998, Owen et al 1998, Andersson et al 1998, Kokkotas & Stergioulas 1998, Lindblom et al 1999), and they have found that for the r -modes the instability due to the gravitational radiation reaction strongly dominates the viscous damping processes considered. For inertial modes, Lockitch & Friedman (1999) suggested that the inertial mode instability due to the gravitational radiation survives the dissipative processes considered for the case of r -modes, but the instability itself is weaker than that of the r -modes. In this section, we reconsider these estimations using a different numerical approach to compute the oscillation modes of a rotating star.

To estimate the dissipation timescales associated with viscosity and gravitational radiation reaction, we employ a simple method used for the analysis of the r -mode instability (Lindblom et al 1998, see also Ipser & Lindblom 1991). When the dissipation timescale is sufficiently long compared to the oscillation period of the mode, the growth rate or the damping rate is approximately evaluated with the non-dissipative eigenfunctions as

$$\frac{1}{\tau} = -\frac{1}{2E} \frac{dE}{dt}, \quad (35)$$

where E is the energy of the oscillation, observed in the corotating frame, given by

$$E = \frac{1}{2} \int \left[\rho \delta v^i \delta v_i^* + \left(\frac{\delta p}{\rho} + \delta \Phi \right) \delta \rho^* \right] d^3x, \quad (36)$$

and the asterisk $*$ denotes the complex conjugate of the indicated quantity. The time derivative of E is determined by the dissipation effects, for which we consider the shear viscosity, bulk viscosity

and gravitational radiation.

The dissipation rate due to the shear viscosity can be calculated from

$$\left(\frac{dE}{dt}\right)_S = -2 \int \eta \delta \sigma^{ij} \delta \sigma_{ij}^* d^3x, \quad (37)$$

where $\delta \sigma_{ij}$ is the shear of the perturbation written by

$$\delta \sigma_{ij} = \frac{1}{2} \left(\nabla_i \delta v_j + \nabla_j \delta v_i - \frac{2}{3} g_{ij} \nabla_k \delta v^k \right), \quad (38)$$

and the coefficient of shear viscosity for hot neutron star matter is given by

$$\eta = 2 \times 10^{18} \left(\frac{\rho}{10^{15} \text{g} \cdot \text{cm}^{-3}} \right)^{\frac{9}{4}} \left(\frac{10^9 K}{T} \right)^2 \text{g} \cdot \text{cm}^{-1} \cdot \text{s}^{-1} \quad (39)$$

(Cutler & Lindblom 1987; Sawyer 1989). The dissipation rate due to the bulk viscosity can be written as

$$\left(\frac{dE}{dt}\right)_B = - \int \zeta \delta \theta \delta \theta^* d^3x, \quad (40)$$

where $\delta \theta$ is the expansion of the perturbation defined as

$$\delta \theta \equiv \nabla_i \delta v^i = -\frac{i\omega}{\Gamma p} \left(\delta p + \xi^a \frac{dp}{da} \right), \quad (41)$$

and the bulk viscosity coefficient for hot neutron star matter is

$$\zeta = 6 \times 10^{25} \left(\frac{1 \text{Hz}}{\sigma + m\Omega} \right)^2 \left(\frac{\rho}{10^{15} \text{g} \cdot \text{cm}^{-3}} \right)^2 \left(\frac{T}{10^9 K} \right)^6 \text{g} \cdot \text{cm}^{-1} \cdot \text{s}^{-1} \quad (42)$$

(Cutler & Lindblom 1987; Sawyer 1989).

The dissipation due to gravitational radiation comes from two divisible contributions, which are

$$\left(\frac{dE}{dt}\right)_{G-D} = -\sigma(\sigma + m\Omega) \sum_{l=2}^{\infty} N_l \sigma^{2l} |\delta D_{lm}|^2, \quad (43)$$

and

$$\left(\frac{dE}{dt}\right)_{G-J} = -\sigma(\sigma + m\Omega) \sum_{l=2}^{\infty} N_l \sigma^{2l} |\delta J_{lm}|^2, \quad (44)$$

where the coupling constant N_l is given by

$$N_l = \frac{4\pi G}{c^{2l+1}} \frac{(l+1)(l+2)}{l(l-1)[(2l+1)!!]^2}. \quad (45)$$

Here, the mass, δD_{lm} , and current, δJ_{lm} , multipole moments of the perturbation are given by (Thorne 1980, Lindblom et al. 1998)

$$\delta D_{lm} = \int \delta \rho r^l Y_l^{*m} d^3x, \quad (46)$$

and

$$\delta J_{lm} = \frac{2}{c} \left(\frac{l}{l+1} \right)^{\frac{1}{2}} \int r^l (\rho \delta v_i + \delta \rho v_i) Y_{lm}^{i,B*} d^3x, \quad (47)$$

and $Y_{lm}^{i,B}$ is the magnetic type vector spherical harmonic given by

$$Y_{lm}^{i,B} = \frac{r}{\sqrt{l(l+1)}} \epsilon^{ijk} \nabla_j Y_l^m \nabla_k r \quad (48)$$

(Thorne 1980). Notice that as indicated by equation (47), the gravitational radiation due to the current multipole moment is emitted by the “axial” parity perturbations.

For inertial modes, since the first term in equation (36) is dominant and the energy E of the modes is positive definite for sufficiently small Ω , the instability sets in when $dE/dt > 0$. As shown by equations (43) and (44), since only the energy change rate due to gravitational radiation can be positive and the energy change rates due to the other dissipative processes are all negative, the necessary condition for the instability is that there exist modes whose frequencies satisfy $\sigma(\sigma + m\Omega) < 0$ and the right-hand-side of equations (43) and (44) becomes positive. Physically, this means that the rotating star becomes unstable when the gravitational radiation carries away negative energy.

We may write the damping timescale of the mode to the lowest order in $\bar{\Omega}$ as follows:

$$\begin{aligned} \frac{1}{\tau} = & \frac{1}{\tilde{\tau}_S} \left(\frac{10^9 K}{T} \right)^2 + \frac{1}{\tilde{\tau}_B} \left(\frac{T}{10^9 K} \right)^6 \left(\frac{\pi G \bar{\rho}}{\Omega^2} \right) \\ & + \sum_{l=2}^{\infty} \frac{1}{\tilde{\tau}_{J,l}} \left(\frac{\Omega^2}{\pi G \bar{\rho}} \right)^{l+1} + \sum_{l=2}^{\infty} \frac{1}{\tilde{\tau}_{D,l}} \left(\frac{\Omega^2}{\pi G \bar{\rho}} \right)^{l+2}, \end{aligned} \quad (49)$$

where $\bar{\rho}$ is the average density of the star. Here, the first, second, third and fourth terms in equation (49) are contributions from the shear viscosity, the bulk viscosity, the current multipole radiation and the mass multipole radiation, respectively. The expression of the timescale τ given by equation (49) is basically the same as that given by Lockitch & Friedman (1999), who ignored the last term, considering that the gravitational radiation from the mass multipole moment is negligible for inertial modes. However, for the even parity modes, the largest components from the mass multipole moment and the current multipole moment may be $\delta D_{|m|,m}$ and $\delta J_{|m|+1,m}$, respectively. If both of the components have the same amplitude, the radiation from the mass multipole moment is by a factor $\bar{\Omega}^{-2}$ larger than that of the current multipole moment. Since the square of the mass multipole moment is by a factor $\bar{\Omega}^2$ smaller than that of the current multipole moment because of the ordering law $\delta \rho \sim O(\bar{\Omega}^2)$ and $\delta v^i \propto \omega \xi^i \sim O(\bar{\Omega})$ given by equation (1), the radiation reaction from the mass multipole is the same order as that from the current multipole for the even parity inertial modes. For consistency, therefore we must include the mass multipole moment as well as the current multipole moment to estimate the energy change rate.

In Table 4, we tabulate the timescales in the unit of second for various dissipative processes for a polytropic model with the index $n = 1$, where the radius and the mass at $\Omega = 0$ are chosen to

be $R = 12.57\text{km}$ and $M = 1.4M_\odot$, respectively. As shown by Table 4, for inertial modes the mass multipole radiation dominates the current multipole radiation for both the even and odd parity modes. Note that for r -modes with $l_0 - |m| = 1$, only the current multipole moment is relevant to the gravitational radiation. We also find that for even parity inertial modes, the fastest growth time due to the mass multipole radiation is of the order of 10^3 seconds for the parameters $10^9 K$ and $\Omega = \sqrt{\pi G \bar{\rho}}$.

Most of the timescales calculated in this paper are in good agreement with those obtained by Lockitch & Friedman (1998) and Lindblom et al. (1999), and the relative difference between the calculations are at most several percents. However, there are a few discrepancies between our calculation and the calculations by Lockitch & Friedman (1998) and Lindblom et al. (1999). We note that the most dominating current multipole δJ_{lm} is that associated with l_{max} and the moment δJ_{lm} with $l < l_{max}$ is less important than $\delta J_{l_{max}m}$, where l_{max} is the largest value of l associated with the dominating expansion coefficients of δv^a . The result that for the odd parity inertial modes the multipole moment δJ_{lm} with $l < l_{max}$ does not vanish may contradict the result obtained by Lockitch & Friedman (1998), who suggested that for the odd parity inertial modes all the current multipole moments δJ_{lm} with $l < l_{max}$ vanish (or nearly vanish). The origin of these discrepancies is not clear at the moment.

As seen from equation (49), the total dissipation timescale τ is given as a function of the angular velocity Ω and the temperature T of a star. For a given temperature T , we can define the critical angular velocity Ω_c such that

$$\frac{1}{\tau(\Omega_c, T)} = 0 \quad \text{for} \quad 0 < \Omega_c < (\pi G \bar{\rho})^{1/2}. \quad (50)$$

With this critical Ω_c , we may say that a rotating neutron star is unstable due to the gravitational radiation when $\Omega > \Omega_c$ because the gravitational radiation reaction dominates the viscous damping processes. In Figure 4, the critical angular rotation velocities Ω_c for two inertial modes and an r -mode are plotted against the temperature of the star. From this figure we can see that the instability of the $m = 2$ even parity inertial mode with $\kappa_0 = 1.100$ is strong, although this instability is weaker than that of the r -mode. At about $T = 10^9 K$ the critical angular velocity of this mode is about 25% of $(\pi G \bar{\rho})^{1/2}$.

6. Discussion and Conclusions

In this paper, we have investigated the properties of inertial mode and r -mode oscillations in rotating isentropic Newtonian stars, using the slow rotation approximation. By taking account of the effects of the rotational deformation of the equilibrium state, we have evaluated the eigenfrequencies of the inertial modes and the r -modes with an accuracy up to the order of $\bar{\Omega}^3$. We have also estimated the dissipation timescales due to the gravitational radiation and the viscosity for a simple neutron star model. We show that the inertial modes emit gravitational

radiation mainly by the mass multipole rather than the current multipole. It is also found that the instability due to the gravitational radiation reaction is strong for the most unstable inertial mode, although the instability associated with the inertial modes is not as strong as that with the r -modes.

In spite of the recent improvements in our understanding about the instability of the inertial modes and the r -modes, it seems that the fundamental properties of these modes have not yet been sufficiently understood, considering that most of the previous investigations of the inertial modes and the r -modes, including the present paper, are restricted to the case of uniformly and slowly rotating, isentropic, Newtonian stars. In this sense, we do not have any clear understanding about the inertial modes and the r -modes of, for example, differentially and rapidly rotating, non-isentropic, relativistic stars. As a first step to relax these restrictions, several studies have already appeared. For example, Andersson, Lockitch & Friedman (1999) suggest a possibility that the assumption of the purely axial mode oscillation leads to inconsistent radial behavior for the case of isentropic *relativistic* stars. Kojima (1998) and Kojima & Hosonuma (1999) also argue that the r -mode spectrum is continuous for relativistic stars. Therefore it is highly desirable to investigate the properties of the inertial modes and the r -modes in the more general case than that investigated previously and in this paper, to conclude that the instability does have an importance in astrophysics.

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A. Basic equations for nonradial oscillations of slowly rotating stars

The derivation of the governing equations of nonradial oscillations of rotating stars is almost the same as that given by Lee & Saio (1986) and Lee (1993), except that the governing equations given in this Appendix are formulated to calculate the eigenfrequencies and eigenfunctions correct up to the order of Ω^3 .

We introduce column vectors \mathbf{y}_1 , \mathbf{y}_2 , \mathbf{y}_3 , \mathbf{y}_4 , \mathbf{h} , and \mathbf{t} , whose components are defined by

$$y_{1,k} = S_l(a), \quad (\text{A1})$$

$$y_{2,k} = \frac{1}{ga} \delta U_l(a) \equiv \frac{1}{ga} \left(\frac{\delta p_l(a)}{\rho} + \delta \Phi_l(a) \right), \quad (\text{A2})$$

$$y_{3,k} = \frac{1}{ga} \delta \Phi_l(a), \quad (\text{A3})$$

$$y_{4,k} = \frac{1}{g} \frac{d\delta \Phi_l(a)}{da}, \quad (\text{A4})$$

$$h_{,k} = H_l(a), \quad (\text{A5})$$

and

$$t_{,k} = T_{l'}(a), \quad (\text{A6})$$

where

$$g = -\frac{1}{\rho} \frac{dp}{da}(a). \quad (\text{A7})$$

Here $l = |m| + 2k - 2$ and $l' = l + 1$ for “even” modes, and $l = |m| + 2k - 1$ and $l' = l - 1$ for “odd” modes, and in both the cases k denote the indices of the vectors, and $k = 1, 2, 3, \dots$

In vector notation, equations of the adiabatic nonradial pulsation for a slowly rotating star are written as follows:

The perturbed mass conservation law (18) reduces to

$$\begin{aligned} a \frac{d\mathbf{y}_1}{da} + \left(3 - \frac{V}{\Gamma}\right) \mathbf{y}_1 + \frac{V}{\Gamma}(\mathbf{y}_2 - \mathbf{y}_3) - \mathbf{\Lambda}_0 \mathbf{h} = & - \left(a \frac{d\vartheta(\alpha)}{da} \mathbf{1} + a \frac{d\vartheta(\beta)}{da} \mathcal{A}_0 \right) \mathbf{y}_1 \\ & + 3\vartheta(\beta) \mathcal{B}_0 \mathbf{h} + 3m\vartheta(\beta) \mathcal{Q}_0 i\mathbf{t}. \end{aligned} \quad (\text{A8})$$

The a component of the perturbed Euler’s equation (19) reduces to

$$\begin{aligned} a \frac{d\mathbf{y}_2}{da} - (c_1 \bar{\omega}^2 + aA_a) \mathbf{y}_1 - (1 - aA_a - U) \mathbf{y}_2 - aA_a \mathbf{y}_3 + 2mc_1 \bar{\omega} \bar{\Omega} \mathbf{h} + 2c_1 \bar{\omega} \bar{\Omega} \mathbf{C}_0 i\mathbf{t} \\ = c_1 \bar{\omega}^2 [2\{\eta(\alpha) \mathbf{1} + \eta(\beta) \mathcal{A}_0\} \mathbf{y}_1 - 3\beta \mathcal{B}_0 \mathbf{h} - 3m\beta \mathcal{Q}_0 i\mathbf{t} \\ - m\nu\{(\alpha + \eta(\alpha)) \mathbf{1} + (\beta + \eta(\beta)) \mathcal{A}_0\} \mathbf{h} \\ - \nu\{(\alpha + \eta(\alpha)) \mathbf{C}_0 + (\beta + \eta(\beta)) \mathcal{E}_0\} i\mathbf{t}]. \end{aligned} \quad (\text{A9})$$

The perturbed Poisson equation (23) reduces to

$$a \frac{d\mathbf{y}_3}{da} - (1 - U) \mathbf{y}_3 - \mathbf{y}_4 = 0, \quad (\text{A10})$$

and

$$\begin{aligned} a \frac{d\mathbf{y}_4}{da} + aA_a U \mathbf{y}_1 - U \frac{V}{\Gamma} \mathbf{y}_2 - \left(\mathbf{\Lambda}_0 - U \frac{V}{\Gamma} \mathbf{1} \right) \mathbf{y}_3 + U \mathbf{y}_4 \\ = 2\{\eta(\alpha) \mathbf{1} + \eta(\beta) \mathcal{A}_0\} a \frac{d\mathbf{y}_4}{da} + \{F(\alpha) \mathbf{1} + F(\beta) \mathcal{A}_0 - 6\beta(\mathcal{A}_0 + \mathcal{B}_0)\} \mathbf{y}_4 \\ - 2\{\alpha \mathbf{1} + \beta \mathcal{A}_0\} \mathbf{\Lambda}_0 \mathbf{y}_3. \end{aligned} \quad (\text{A11})$$

Here,

$$U = \frac{d \ln M(a)}{d \ln a}, \quad V = -\frac{d \ln p}{d \ln a}, \quad (\text{A12})$$

$$\vartheta(\alpha) = 3\alpha + a \frac{d\alpha}{da}, \quad \eta(\alpha) = \alpha + a \frac{d\alpha}{da}, \quad (\text{A13})$$

$$F(\alpha) = 2U\eta(\alpha) - a \frac{d\alpha}{da} + a \frac{d}{da} \left(a \frac{d\alpha}{da} \right), \quad (\text{A14})$$

$\bar{\omega} \equiv \omega/(GM/R^3)^{1/2}$ is the frequency in the unit of the Kepler frequency, and $\nu \equiv 2\Omega/\omega$.

The θ and φ components of the perturbed Euler's equation (19) reduce to

$$\tilde{\mathbf{L}}_0 \mathbf{h} - \tilde{\mathbf{M}}_1 i\mathbf{t} = \frac{1}{c_1 \bar{\omega}^2} \mathbf{y}_2 + \{\tilde{\mathbf{O}} + \tilde{\mathbf{M}}_1 \tilde{\mathbf{L}}_1^{-1} \tilde{\mathbf{K}}\} \mathbf{y}_1, \quad (\text{A15})$$

$$\tilde{\mathbf{L}}_1 i\mathbf{t} - \tilde{\mathbf{M}}_0 \mathbf{h} = -\tilde{\mathbf{K}} \mathbf{y}_1, \quad (\text{A16})$$

where

$$\tilde{\mathbf{L}}_0 = (1 + 2\alpha)\mathbf{L}_0 + \beta\mathbf{\Lambda}_0^{-1}\mathcal{H}_0 - 2m\nu\beta\mathbf{\Lambda}_0^{-1}(6\mathcal{A}_0 + \mathbf{1}), \quad (\text{A17})$$

$$\tilde{\mathbf{L}}_1 = (1 + 2\alpha)\mathbf{L}_1 + \beta\mathbf{\Lambda}_1^{-1}\mathcal{H}_1 - 2m\nu\beta\mathbf{\Lambda}_1^{-1}(6\mathcal{A}_1 + \mathbf{1}), \quad (\text{A18})$$

$$\tilde{\mathbf{M}}_0 = (1 + 2\alpha - 2\beta)\mathbf{M}_0 - 6m\beta\mathbf{\Lambda}_1^{-1}\mathcal{Q}_1 + 4\nu\beta\mathbf{\Lambda}_1^{-1}(\mathcal{D}_1\mathbf{\Lambda}_0 + 3\mathcal{E}_1 + \mathbf{C}_1), \quad (\text{A19})$$

$$\tilde{\mathbf{M}}_1 = (1 + 2\alpha - 2\beta)\mathbf{M}_1 - 6m\beta\mathbf{\Lambda}_0^{-1}\mathcal{Q}_0 + 4\nu\beta\mathbf{\Lambda}_0^{-1}(\mathcal{D}_0\mathbf{\Lambda}_1 + 3\mathcal{E}_0 + \mathbf{C}_0), \quad (\text{A20})$$

$$\tilde{\mathbf{K}} = (1 + \alpha + \eta(\alpha))\nu\mathbf{K} - 3m\beta\mathbf{\Lambda}_1^{-1}\mathcal{Q}_1 + \nu(\beta + \eta(\beta))\mathbf{\Lambda}_1^{-1}(4\mathcal{D}_1 + \mathcal{E}_1 - 2\mathcal{Q}_1), \quad (\text{A21})$$

and

$$\tilde{\mathbf{O}} = \mathbf{\Lambda}_1^{-1} \left[m\nu \left\{ (1 + \alpha + \eta(\alpha)) \mathbf{1} + (\beta + \eta(\beta)) \mathcal{A}_0 \right\} - 3\beta(2\mathcal{A}_0 + \mathcal{B}_0) \right] - \tilde{\mathbf{M}}_1 \tilde{\mathbf{L}}_1^{-1} \tilde{\mathbf{K}}; \quad (\text{A22})$$

$$\mathcal{H}_0 = 2\mathcal{A}_0\mathbf{\Lambda}_0 + 6\mathcal{B}_0, \quad \mathcal{H}_1 = 2\mathcal{A}_1\mathbf{\Lambda}_1 + 6\mathcal{B}_1. \quad (\text{A23})$$

The quantities \mathcal{Q}_0 , \mathcal{Q}_1 , \mathbf{C}_0 , \mathbf{C}_1 , \mathbf{K} , \mathbf{L}_0 , \mathbf{L}_1 , $\mathbf{\Lambda}_0$, $\mathbf{\Lambda}_1$, \mathbf{M}_0 , \mathbf{M}_1 , \mathcal{A}_0 , \mathcal{A}_1 , \mathcal{B}_0 , \mathcal{B}_1 , \mathcal{D}_0 , \mathcal{D}_1 , \mathcal{E}_0 , and \mathcal{E}_1 are matrices written as follows:

For even modes,

$$\begin{aligned} (\mathcal{Q}_0)_{i,i} &= J_{l+1}^m, & (\mathcal{Q}_0)_{i+1,i} &= J_{l+2}^m, \\ (\mathcal{Q}_1)_{i,i} &= J_{l+1}^m, & (\mathcal{Q}_1)_{i,i+1} &= J_{l+2}^m, \\ (\mathbf{C}_0)_{i,i} &= -(l+2)J_{l+1}^m, & (\mathbf{C}_0)_{i+1,i} &= (l+1)J_{l+2}^m, \\ (\mathbf{C}_1)_{i,i} &= lJ_{l+1}^m, & (\mathbf{C}_1)_{i,i+1} &= -(l+3)J_{l+2}^m, \\ (\mathbf{K})_{i,i} &= \frac{J_{l+1}^m}{l+1}, & (\mathbf{K})_{i,i+1} &= -\frac{J_{l+2}^m}{l+2}, \\ (\mathbf{L}_0)_{i,i} &= 1 - \frac{m\nu}{l(l+1)}, & (\mathbf{L}_1)_{i,i} &= 1 - \frac{m\nu}{(l+1)(l+2)}, \\ (\mathbf{\Lambda}_0)_{i,i} &= l(l+1), & (\mathbf{\Lambda}_1)_{i,i} &= (l+1)(l+2), \\ (\mathbf{M}_0)_{i,i} &= \nu \frac{l}{l+1} J_{l+1}^m, & (\mathbf{M}_0)_{i,i+1} &= \nu \frac{l+3}{l+2} J_{l+2}^m, \\ (\mathbf{M}_1)_{i,i} &= \nu \frac{l+2}{l+1} J_{l+1}^m, & (\mathbf{M}_1)_{i+1,i} &= \nu \frac{l+1}{l+2} J_{l+2}^m, \\ (\mathcal{A}_0)_{i,i} &= -\frac{1}{2} + \frac{3}{2} \{(J_l^m)^2 + (J_{l+1}^m)^2\}, & (\mathcal{A}_0)_{i+1,i} &= (\mathcal{A}_0)_{i,i+1} = \frac{3}{2} J_{l+1}^m J_{l+2}^m, \end{aligned}$$

$$\begin{aligned}
(\mathcal{A}_1)_{i,i} &= -\frac{1}{2} + \frac{3}{2} \{(J_{l+2}^m)^2 + (J_{l+1}^m)^2\}, & (\mathcal{A}_1)_{i+1,i} &= (\mathcal{A}_1)_{i,i+1} = \frac{3}{2} J_{l+2}^m J_{l+3}^m, \\
(\mathcal{B}_0)_{i,i} &= l(J_{l+1}^m)^2 - (l+1)(J_l^m)^2, & (\mathcal{B}_0)_{i+1,i} &= lJ_{l+1}^m J_{l+2}^m, \\
(\mathcal{B}_0)_{i,i+1} &= -(l+3)J_{l+1}^m J_{l+2}^m, & (\mathcal{B}_1)_{i,i} &= (l+1)(J_{l+2}^m)^2 - (l+2)(J_{l+1}^m)^2, \\
(\mathcal{B}_1)_{i+1,i} &= (l+1)J_{l+2}^m J_{l+3}^m, & (\mathcal{B}_1)_{i,i+1} &= -(l+4)J_{l+2}^m J_{l+3}^m, \\
(\mathcal{D}_0)_{i,i} &= J_{l+1}^m \left[-\frac{1}{2} + \frac{3}{2} \{(J_l^m)^2 + (J_{l+1}^m)^2 + (J_{l+2}^m)^2\} \right], & (\mathcal{D}_0)_{i,i+1} &= \frac{3}{2} J_{l+1}^m J_{l+2}^m J_{l+3}^m, \\
(\mathcal{D}_0)_{i+1,i} &= J_{l+2}^m \left[-\frac{1}{2} + \frac{3}{2} \{(J_{l+1}^m)^2 + (J_{l+2}^m)^2 + (J_{l+3}^m)^2\} \right], & (\mathcal{D}_0)_{i+2,i} &= \frac{3}{2} J_{l+2}^m J_{l+3}^m J_{l+4}^m, \\
(\mathcal{D}_1)_{i,i} &= J_{l+1}^m \left[-\frac{1}{2} + \frac{3}{2} \{(J_l^m)^2 + (J_{l+1}^m)^2 + (J_{l+2}^m)^2\} \right], & (\mathcal{D}_1)_{i+1,i} &= \frac{3}{2} J_{l+1}^m J_{l+2}^m J_{l+3}^m, \\
(\mathcal{D}_1)_{i,i+1} &= J_{l+2}^m \left[-\frac{1}{2} + \frac{3}{2} \{(J_{l+1}^m)^2 + (J_{l+2}^m)^2 + (J_{l+3}^m)^2\} \right], & (\mathcal{D}_1)_{i,i+2} &= \frac{3}{2} J_{l+2}^m J_{l+3}^m J_{l+4}^m, \\
(\mathcal{E}_0)_{i,i} &= J_{l+1}^m \left[\frac{1}{2} (l+2) + \frac{3}{2} \{(l+1)(J_{l+2}^m)^2 - (l+2)((J_l^m)^2 + (J_{l+1}^m)^2)\} \right], \\
(\mathcal{E}_0)_{i+1,i} &= J_{l+2}^m \left[-\frac{1}{2} (l+1) + \frac{3}{2} \{(l+1)((J_{l+2}^m)^2 + (J_{l+3}^m)^2) - (l+2)(J_{l+1}^m)^2\} \right], \\
(\mathcal{E}_0)_{i,i+1} &= -\frac{3}{2} (l+4)J_{l+1}^m J_{l+2}^m J_{l+3}^m, & (\mathcal{E}_0)_{i+2,i} &= \frac{3}{2} (l+1)J_{l+2}^m J_{l+3}^m J_{l+4}^m, \\
(\mathcal{E}_1)_{i,i} &= J_{l+1}^m \left[-\frac{1}{2} l + \frac{3}{2} \{l((J_{l+1}^m)^2 + (J_{l+2}^m)^2) - (l+1)(J_l^m)^2\} \right], \\
(\mathcal{E}_1)_{i,i+1} &= J_{l+2}^m \left[\frac{1}{2} (l+3) + \frac{3}{2} \{(l+2)(J_{l+3}^m)^2 - (l+3)((J_{l+1}^m)^2 + (J_{l+2}^m)^2)\} \right], \\
(\mathcal{E}_1)_{i+1,i} &= \frac{3}{2} lJ_{l+1}^m J_{l+2}^m J_{l+3}^m, & (\mathcal{E}_1)_{i,i+2} &= -\frac{3}{2} (l+5)J_{l+2}^m J_{l+3}^m J_{l+4}^m,
\end{aligned}
\tag{A24}$$

where $l = |m| + 2i - 2$, $i = 1, 2, 3, \dots$, and

$$J_l^m \equiv \left[\frac{(l+m)(l-m)}{(2l-1)(2l+1)} \right]^{\frac{1}{2}}. \tag{A25}$$

For odd modes,

$$\begin{aligned}
(\mathcal{Q}_0)_{i,i} &= J_l^m, & (\mathcal{Q}_0)_{i,i+1} &= J_{l+1}^m, \\
(\mathcal{Q}_1)_{i,i} &= J_l^m, & (\mathcal{Q}_1)_{i+1,i} &= J_{l+1}^m, \\
(\mathbf{C}_0)_{i,i} &= (l-1)J_l^m, & (\mathbf{C}_0)_{i+1,i} &= -(l+2)J_{l+1}^m, \\
(\mathbf{C}_1)_{i,i} &= -(l+1)J_l^m, & (\mathbf{C}_1)_{i+1,i} &= lJ_{l+1}^m, \\
(\mathbf{K})_{i,i} &= -\frac{J_l^m}{l}, & (\mathbf{K})_{i+1,i} &= \frac{J_{l+1}^m}{l+1},
\end{aligned}$$

$$\begin{aligned}
(\mathbf{L}_0)_{i,i} &= 1 - \frac{m\nu}{l(l+1)}, & (\mathbf{L}_1)_{i,i} &= 1 - \frac{m\nu}{l(l-1)}, \\
(\mathbf{\Lambda}_0)_{i,i} &= l(l+1), & (\mathbf{\Lambda}_1)_{i,i} &= l(l-1), \\
(\mathbf{M}_0)_{i,i} &= \nu \frac{l+1}{l} J_l^m, & (\mathbf{M}_0)_{i+1,i} &= \nu \frac{l}{l+1} J_{l+1}^m, \\
(\mathbf{M}_1)_{i,i} &= \nu \frac{l-1}{l} J_l^m, & (\mathbf{M}_1)_{i,i+1} &= \nu \frac{l+2}{l+1} J_{l+1}^m, \\
(\mathcal{A}_0)_{i,i} &= -\frac{1}{2} + \frac{3}{2} \{(J_{l+1}^m)^2 + (J_l^m)^2\}, & (\mathcal{A}_0)_{i+1,i} &= (\mathcal{A}_0)_{i,i+1} = \frac{3}{2} J_{l+1}^m J_{l+2}^m, \\
(\mathcal{A}_1)_{i,i} &= -\frac{1}{2} + \frac{3}{2} \{(J_{l-1}^m)^2 + (J_l^m)^2\}, & (\mathcal{A}_1)_{i+1,i} &= (\mathcal{A}_1)_{i,i+1} = \frac{3}{2} J_l^m J_{l+1}^m, \\
(\mathcal{B}_0)_{i,i} &= l(J_{l+1}^m)^2 - (l+1)(J_l^m)^2, & (\mathcal{B}_0)_{i+1,i} &= lJ_{l+1}^m J_{l+2}^m, \\
(\mathcal{B}_0)_{i,i+1} &= -(l+3)J_{l+1}^m J_{l+2}^m, & (\mathcal{B}_1)_{i,i} &= (l-1)(J_l^m)^2 - l(J_{l-1}^m)^2, \\
(\mathcal{B}_1)_{i+1,i} &= (l-1)J_l^m J_{l+1}^m, & (\mathcal{B}_1)_{i,i+1} &= -(l+2)J_l^m J_{l+1}^m, \\
(\mathcal{D}_0)_{i,i} &= J_l^m \left[-\frac{1}{2} + \frac{3}{2} \{(J_{l-1}^m)^2 + (J_l^m)^2 + (J_{l+1}^m)^2\} \right], & (\mathcal{D}_0)_{i+1,i} &= \frac{3}{2} J_l^m J_{l+1}^m J_{l+2}^m, \\
(\mathcal{D}_0)_{i,i+1} &= J_{l+1}^m \left[-\frac{1}{2} + \frac{3}{2} \{(J_l^m)^2 + (J_{l+1}^m)^2 + (J_{l+2}^m)^2\} \right], & (\mathcal{D}_0)_{i,i+2} &= \frac{3}{2} J_{l+1}^m J_{l+2}^m J_{l+3}^m, \\
(\mathcal{D}_1)_{i,i} &= J_l^m \left[-\frac{1}{2} + \frac{3}{2} \{(J_{l-1}^m)^2 + (J_l^m)^2 + (J_{l+1}^m)^2\} \right], & (\mathcal{D}_1)_{i,i+1} &= \frac{3}{2} J_l^m J_{l+1}^m J_{l+2}^m, \\
(\mathcal{D}_1)_{i+1,i} &= J_{l+1}^m \left[-\frac{1}{2} + \frac{3}{2} \{(J_l^m)^2 + (J_{l+1}^m)^2 + (J_{l+2}^m)^2\} \right], & (\mathcal{D}_1)_{i+2,i} &= \frac{3}{2} J_{l+1}^m J_{l+2}^m J_{l+3}^m, \\
(\mathcal{E}_0)_{i,i} &= J_l^m \left[-\frac{1}{2}(l-1) + \frac{3}{2} \{(l-1)((J_l^m)^2 + (J_{l+1}^m)^2) - l(J_{l-1}^m)^2\} \right], \\
(\mathcal{E}_0)_{i,i+1} &= J_{l+1}^m \left[\frac{1}{2}(l+2) + \frac{3}{2} \{-(l+2)((J_l^m)^2 + (J_{l+1}^m)^2) + (l+1)(J_{l+2}^m)^2\} \right], \\
(\mathcal{E}_0)_{i+1,i} &= \frac{3}{2} (l-1) J_l^m J_{l+1}^m J_{l+2}^m, & (\mathcal{E}_0)_{i,i+2} &= -\frac{3}{2} (l+4) J_{l+1}^m J_{l+2}^m J_{l+3}^m, \\
(\mathcal{E}_1)_{i,i} &= J_l^m \left[\frac{1}{2} (l+1) + \frac{3}{2} \{l(J_{l+1}^m)^2 - (l+1)((J_{l-1}^m)^2 + (J_l^m)^2)\} \right], \\
(\mathcal{E}_1)_{i+1,i} &= J_{l+1}^m \left[-\frac{1}{2} l + \frac{3}{2} \{l((J_{l+1}^m)^2 + (J_{l+2}^m)^2) - (l+1)(J_l^m)^2\} \right], \\
(\mathcal{E}_1)_{i,i+1} &= -\frac{3}{2} (l+3) J_l^m J_{l+1}^m J_{l+2}^m, & (\mathcal{E}_1)_{i+2,i} &= \frac{3}{2} l J_{l+1}^m J_{l+2}^m J_{l+3}^m,
\end{aligned} \tag{A26}$$

where $l = |m| + 2i - 1$, $i = 1, 2, 3, \dots$

Eliminating \mathbf{h} and $i\mathbf{t}$ from equations (A8) and (A9) by using equations (A15) and (A16), equations (A8)–(A11) reduce to a set of first-order linear ordinary differential equations for \mathbf{y}_1 , \mathbf{y}_2 , \mathbf{y}_3 and \mathbf{y}_4 as follows:

$$a \frac{d\mathbf{y}_1}{da} = \left\{ \left(\frac{V}{\Gamma} - 3 \right) \mathbf{1} + \mathcal{F}_{11} \right\} \mathbf{y}_1 + \left(\frac{\mathcal{F}_{12}}{c_1 \bar{\omega}^2} - \frac{V}{\Gamma} \mathbf{1} \right) \mathbf{y}_2 + \frac{V}{\Gamma} \mathbf{y}_3, \quad (\text{A27})$$

$$a \frac{d\mathbf{y}_2}{da} = \left\{ (c_1 \bar{\omega}^2 + aA_a) \mathbf{1} + c_1 \bar{\omega}^2 \mathcal{F}_{21} \right\} \mathbf{y}_1 + \left\{ (1 - aA_a - U) \mathbf{1} + \mathcal{F}_{22} \right\} \mathbf{y}_2 + aA_a \mathbf{y}_3, \quad (\text{A28})$$

$$a \frac{d\mathbf{y}_3}{da} = (1 - U) \mathbf{y}_3 + \mathbf{y}_4, \quad (\text{A29})$$

$$\begin{aligned} a \frac{d\mathbf{y}_4}{da} = & -aA_a U \mathcal{J}^{-1} \mathbf{y}_1 + U \frac{V}{\Gamma} \mathcal{J}^{-1} \mathbf{y}_2 + \mathcal{J}^{-1} \left\{ \mathbf{\Lambda}_0 - U \frac{V}{\Gamma} \mathbf{1} - 2(\alpha \mathbf{1} + \beta \mathcal{A}_0) \mathbf{\Lambda}_0 \right\} \mathbf{y}_3 \\ & + \mathcal{J}^{-1} \left\{ -U \mathbf{1} + F(\alpha) \mathbf{1} + F(\beta) \mathcal{A}_0 - 6\beta(\mathcal{A}_0 + \mathcal{B}_0) \right\} \mathbf{y}_4, \end{aligned} \quad (\text{A30})$$

where

$$\mathcal{F}_{11} = \left\{ \mathbf{1} + 3\vartheta(\beta) \mathcal{P} \right\} \tilde{\mathbf{W}} \tilde{\mathbf{O}} - a \frac{d\vartheta(\alpha)}{da} \mathbf{1} - a \frac{d\vartheta(\beta)}{da} \mathcal{A}_0 - 3m\vartheta(\beta) \mathcal{Q}_0 \tilde{\mathbf{L}}_1^{-1} \tilde{\mathbf{K}}, \quad (\text{A31})$$

$$\mathcal{F}_{12} = \left\{ \mathbf{1} + 3\vartheta(\beta) \mathcal{P} \right\} \tilde{\mathbf{W}}, \quad (\text{A32})$$

$$\begin{aligned} \mathcal{F}_{21} = & 2 \left\{ \eta(\alpha) \mathbf{1} + \eta(\beta) \mathcal{A}_0 \right\} - \mathcal{R} \tilde{\mathbf{W}} \tilde{\mathbf{O}} \\ & + \left[\nu \left\{ (1 + \alpha + \eta(\alpha)) \mathbf{C}_0 + (\beta + \eta(\beta)) \mathcal{E}_0 \right\} + 3m\beta \mathcal{Q}_0 \right] \tilde{\mathbf{L}}_1^{-1} \tilde{\mathbf{K}}, \end{aligned} \quad (\text{A33})$$

$$\mathcal{F}_{22} = -\mathcal{R} \tilde{\mathbf{W}}, \quad (\text{A34})$$

$$\mathcal{P} = \left(\mathcal{B}_0 + m \mathcal{Q}_0 \tilde{\mathbf{L}}_1^{-1} \tilde{\mathbf{M}}_0 \right) \mathbf{\Lambda}_0^{-1}, \quad (\text{A35})$$

$$\begin{aligned} \mathcal{R} = & \left[m\nu \left\{ (1 + \alpha + \eta(\alpha)) \mathbf{1} + (\beta + \eta(\beta)) \mathcal{A}_0 \right\} + 3\beta \mathcal{B}_0 \right] \mathbf{\Lambda}_0^{-1} \\ & + \left[\nu \left\{ (1 + \alpha + \eta(\alpha)) \mathbf{C}_0 + (\beta + \eta(\beta)) \mathcal{E}_0 + 3m\beta \mathcal{Q}_0 \right\} \right] \tilde{\mathbf{L}}_1^{-1} \tilde{\mathbf{M}}_0 \mathbf{\Lambda}_0^{-1}, \end{aligned} \quad (\text{A36})$$

$$\tilde{\mathbf{W}} = \mathbf{\Lambda}_0 (\tilde{\mathbf{L}}_0 - \tilde{\mathbf{M}}_1 \tilde{\mathbf{L}}_1^{-1} \tilde{\mathbf{M}}_0)^{-1}, \quad (\text{A37})$$

$$\mathcal{J} = \mathbf{1} - 2 \left\{ \eta(\alpha) \mathbf{1} + \eta(\beta) \mathcal{A}_0 \right\}. \quad (\text{A38})$$

The surface boundary conditions are

$$\mathbf{y}_1 - \mathbf{y}_2 + \mathbf{y}_3 = 0, \quad (\text{A39})$$

and

$$\left[\mathbf{\Lambda}_2 \left\{ (1 + \alpha) \mathbf{1} + \beta \mathcal{A}_0 \right\} + \left\{ \alpha \mathbf{1} + \beta \mathcal{A}_0 \right\} \mathbf{\Lambda}_0 \right] \mathbf{y}_3 + \left\{ \mathbf{\Lambda}_2 (\alpha \mathbf{1} + \beta \mathcal{A}_0) + \mathbf{1} \right\} \mathbf{y}_4 = 0, \quad (\text{A40})$$

where $\mathbf{\Lambda}_2$ is a matrix whose elements are given by

$$(\mathbf{\Lambda}_2)_{i,i} = (l + 1). \quad (\text{A41})$$

Here $l = |m| + 2i - 2$ for “even” modes, $l = |m| + 2i - 1$ for “odd” modes. The inner boundary conditions at the stellar center are the regularity conditions of the eigenfunctions.

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Fig. 1.— The first two expansion coefficients $S_l(a)$ [panel a], $H_l(a)$ [panel b], $iT_l(a)$ [panel c] for the even parity inertial modes with $m = 2$ and $l_0 - |m| = 2$ are plotted against a/R for a polytropic model with the index $n = 1$, where we assume $\bar{\Omega} = 0.01$. The solid curves give the expansion coefficients corresponding to the mode with $\kappa_0 = 1.100$, while the dotted curves correspond to the mode having $\kappa_0 = -0.557$. The eigenfunctions are normalized so that $iT_3(a = R) = 1$. Attached labels $l = 2, 3, 4, 5, \dots$ denote the spherical harmonic indices associated with the expansion coefficients.

Fig. 2.— The first three expansion coefficients $S_l(a)$ [panel a], $H_l(a)$ [panel b], $iT_l(a)$ [panel c] for the $m = 2$ even parity inertial mode with $\kappa_0 = 1.520$ and $l_0 - |m| = 4$ are plotted against a/R for a polytropic model with the index $n = 1$, where $\bar{\Omega} = 0.01$. The eigenfunctions are normalized so that $iT_3(a = R) = 1$. Attached labels $l = 2, 3, 4, 5, \dots$ denote the spherical harmonic indices associated with the expansion coefficients.

Fig. 3.— The same as Figure 2, but for the mode having eigenvalue $\kappa_0 = 0.863$.

Fig. 4.— Critical angular velocities Ω_c for the r -mode and two inertial modes are plotted against the temperature of a neutron star. The curve labeled “ r -mode” is for the $m = 2$ r -mode with $\kappa_0 = 0.667$. The curve labeled “even” is for the $m = 2$ even parity inertial mode with $\kappa_0 = 1.100$. The curve labeled “odd” is for the $m = 2$ odd parity inertial mode with $\kappa_0 = 0.517$. Here, the critical angular velocity is normalized by using $(\pi G \bar{\rho})^{1/2}$ and the unit of the temperature is Kelvin.

Table 1. Eigenvalues $(\kappa_0, \kappa_2)^a$ of inertial modes and r -modes for the $p = K\rho^2$ Polytrope.

$l_0 - m ^b$	$m = 1$	$m = 2$	$m = 3$
1 ^c	(1.00000, -0.00018)	(0.66667, 0.39827)*	(0.50000, 0.42727)*
2	(-0.40137, 0.20020)	(-0.55659, 0.12248)	(-0.63164, 0.05993)
	(1.41300, 0.23767)	(1.10003, 0.46560)*	(0.90491, 0.54494)*
3	(-1.03238, -0.21090)	(-1.02588, -0.22877)	(-1.01487, -0.23750)
	(0.69059, 0.36221)*	(0.51734, 0.37950)*	(0.41265, 0.35766)*
	(1.61373, 0.26999)	(1.35778, 0.45492)*	(1.17674, 0.55092)*
4	(-1.31227, -0.32577)	(-1.27289, -0.33970)	(-1.23863, -0.34435)
	(-0.17879, 0.01402)	(-0.27533, 0.00051)	(-0.33327, -0.01537)
	(1.05153, 0.45459)	(0.86295, 0.50140)*	(0.73430, 0.50562)*
	(1.72626, 0.24391)	(1.51957, 0.41210)*	(1.36056, 0.51593)*

^a $\Omega(\kappa_0 + \kappa_2\bar{\Omega}^2) = \omega$ is the mode frequency in the corotating frame up to the order of $\bar{\Omega}^3$. The modes marked with an asterisk * satisfy the condition $\sigma(\sigma + m\Omega) < 0$ to the lowest order in Ω , and are considered to be unstable to the gravitational radiation reaction in the absence of viscous dissipation.

^bOur definition of the angular quantum number l_0 is not the same as that of Lockitch & Friedman (1998), but the same as that of Lindblom & Ipser (1998). Eigenmodes having odd (even) values of $l_0 - |m|$ are odd (even) parity modes.

^cThese modes are r -modes. To the lowest order in Ω , $\kappa_0 = 2/(m+1)$ (Papalouizou & Pringle 1978) and is independent of the equation of state.

Table 2. Eigenvalues $(\kappa_0, \kappa_2)^a$ of $m = 2$ inertial modes and r -modes for several values of the polytropic index n .

$l_0 - m ^b$	$n = 0$	$n = 1$	$n = 1.5$	$n = 2$
1 ^c	(0.66667, 0.76543)*	(0.66667, 0.39827)*	(0.66667, 0.28248)*	(0.66667, 0.19701)*
2	(-0.23193, 0.02930)	(-0.55659, 0.12248)	(-0.69646, 0.13213)	(-0.82771, 0.12773)
	(1.23193, 0.88145)*	(1.10003, 0.46560)*	(1.06259, 0.33590)*	(1.03438, 0.24322)*
3	(-0.76334,-0.55320)	(-1.02588,-0.22877)	(-1.12777,-0.15318)	(-1.21712,-0.11767)
	(0.46690, 0.62847)*	(0.51734, 0.37950)*	(0.53563, 0.29081)*	(0.55163, 0.22228)*
	(1.49644, 0.77599)*	(1.35778, 0.45492)*	(1.31003, 0.34136)*	(1.27054, 0.25749)*
4	(-1.09257,-0.77973)	(-1.27289,-0.33970)	(-1.34194,-0.24015)	(-1.40134,-0.18109)
	(-0.10179, 0.00439)	(-0.27533, 0.00051)	(-0.36424,-0.01189)	(-0.45689,-0.02235)
	(0.88425, 0.89693)*	(0.86295, 0.50140)*	(0.85863, 0.38371)*	(0.85688, 0.29910)*
	(1.64344, 0.63981)*	(1.51957, 0.41210)*	(1.47219, 0.32232)*	(1.43071, 0.25292)*

^a $\Omega(\kappa_0 + \kappa_2 \bar{\Omega}^2) = \omega$ is the mode frequency in the corotating frame up to the order of $\bar{\Omega}^3$. The modes marked with an asterisk * satisfy the condition $\sigma(\sigma + m\Omega) < 0$ to the lowest order in Ω , and are considered to be unstable to the gravitational radiation reaction the absence of viscous dissipation.

^bOur definition of the angular quantum number l_0 is not the same as that of Lockitch & Friedman (1998), but the same as that of Lindblom & Ipser (1998). Eigenmodes having odd (even) values of $l_0 - |m|$ are odd (even) parity modes.

^cThese modes are r -modes. To the lowest order in Ω , $\kappa_0 = 2/(m + 1)$ (Papalouizou & Prigle 1978) and is independent of the equation of state.

Table 3. Node number of the dominating expansion coefficients of δv^i for r -modes and inertial modes.

$l_0 - m $ ^a	1	2	3	4
node number of $S_{ m }$ ^b	...	1	...	2
node number of $S_{ m +1}$	1	...
node number of $S_{ m +2}$	1
node number of $S_{ m +3}$
node number of $H_{ m }$...	1	...	2
node number of $H_{ m +1}$	1	...
node number of $H_{ m +2}$	1
node number of $H_{ m +3}$
node number of $iT_{ m }$	0	...	1	...
node number of $iT_{ m +1}$...	0	...	1
node number of $iT_{ m +2}$	0	...
node number of $iT_{ m +3}$	0

^aOur definition of angular quantum number l_0 is not the same as that of Lockitch and Friedman (1998), but that of Lindblom and Ipser (1998). Eigenmodes having odd (even) values of $l_0 - |m|$ are corresponding to odd (even) parity modes.

^bFor expansion coefficients S_l , we include the node at the stellar surface in the count of nodes.

Table 4. Dissipative timescales of inertial modes^a and r -modes for $n = 1$ polytropic model at $T = 10^9 K$ and $\Omega = \sqrt{\pi G \bar{\rho}}$.

m	$l_0 - m $	κ_0	$\tilde{\tau}_B(\text{s})$	$\tilde{\tau}_S(\text{s})$	$\tilde{\tau}_2(\text{s})^b$	$\tilde{\tau}_3(\text{s})$	$\tilde{\tau}_4(\text{s})$	$\tilde{\tau}_5(\text{s})$
1	3	0.691	5.89×10^9	9.23×10^7	$-2.46 \times 10^5 \dagger$	-1.27×10^8
2	1 ^c	0.667	2.03×10^{11}	2.50×10^8	-3.31×10^0	$-3.49 \times 10^2 \dagger$
	2	1.100	3.35×10^9	1.23×10^8	$-1.71 \times 10^3 \dagger$	-3.43×10^4
	3	0.517	6.47×10^9	6.18×10^7	-1.31×10^8	$-8.39 \times 10^4 \dagger$	-1.88×10^6	...
		1.358	4.13×10^9	7.14×10^7	-7.12×10^9	$-1.69 \times 10^7 \dagger$	-1.63×10^9	...
	4	0.863	1.94×10^9	4.92×10^7	$-1.63 \times 10^5 \dagger$	-1.12×10^7
		1.520	4.82×10^9	4.87×10^7	$-3.35 \times 10^7 \dagger$	-1.90×10^{11}
3	1 ^c	0.500	6.63×10^{10}	1.43×10^8	...	-3.17×10^1	$-1.88 \times 10^3 \dagger$...
	2	0.905	1.88×10^9	9.41×10^7	...	$-8.62 \times 10^3 \dagger$	-2.77×10^4	...
	3	0.413	6.97×10^9	4.78×10^7	...	-1.86×10^{10}	$-5.30 \times 10^5 \dagger$	-4.07×10^6
		1.177	2.49×10^9	6.11×10^7	...	-1.65×10^{11}	$-5.11 \times 10^6 \dagger$	-6.11×10^7
	4	0.734	1.29×10^9	4.10×10^7	...	$-2.00 \times 10^6 \dagger$	-1.16×10^7	...
		1.361	3.06×10^9	4.18×10^7	...	$-3.45 \times 10^7 \dagger$	-1.75×10^{10}	...

^aWe present dissipative timescales only for those modes that are unstable to gravitational radiation reaction.

^bWe present dissipative timescales $\tilde{\tau}_l$ due to the gravitational radiation reaction only for those of the dominant multipole moments, where for modes with odd l_0 $\tilde{\tau}_{2k+1} = \tilde{\tau}_{D,2k+1}$ and $\tilde{\tau}_{2k} = \tilde{\tau}_{J,2k}$, and for modes with even l_0 $\tilde{\tau}_{2k+1} = \tilde{\tau}_{J,2k+1}$ and $\tilde{\tau}_{2k} = \tilde{\tau}_{D,2k}$. Here $k = 1, 2, 3, \dots$. Dissipative timescales due to the mass multipole radiation are marked with a dagger \dagger .

^cThis mode is the r -mode. Since the expansion of perturbations vanishes to the lowest order in Ω for r -modes, the definition of the bulk viscosity timescale $\tilde{\tau}_B$ for this mode is not the same as that of equation (49), but that of Lindblom et al. (1999).



















